Inroduction to Linear Quadratic Optimal Control

MEM 355 Performance Enhancement of Dynamical Systems

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Outline

- State Feedback Regulators
  - The LQR Problem
  - Solving the Riccati Equation
- Output Feedback Regulation
  - Coping with Measurement Noise & Disturbances
  - Classical LQG

Regulation: Steer the system from any initial state to the origin.

The Linear Quadratic Regulator Problem (state feedback)

Given : $\dot{x} = Ax + Bu$, Find a state feedback control $u = Kx$

that minimizes the cost function

$$J(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau) \right] d\tau$$

<u>Theorem</u>: If (A, B) is controllable (stabilizable) the optimal controller

for any initial state is $u(t) = Kx(t)$, where:

$$K = -R^{-1}B^T X$$

and $X = X^T \geq 0$ is the unique positive semidefinite solution

of the algebraic Riccati equation:

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

Stability of LQR~1

Show that the optimal state feedback control results in a stable closed loop system.

An autonomous linear system $\dot{x} = Ax$ is asymptotically stable if there exists $P > 0, Q > 0$ that satisfy the Lyapunov equation

$$PA + A^T P = -Q$$

Furthermore, if $\det A \neq 0$, the system is asymptotically if and only if there exist, $P > 0, Q \geq 0$ that satisfy the Lyapunov equation.

Basis of proof: set $V(x) = x^T Px,\ P > 0$ and compute

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = x^T PAx + x^T A^T Px = x^T\left(PA + A^T P\right)x$$

If $\dot{V} = -x^T Qx,\ Q > 0$ we have stability by the Lyapunov stability Theorem. If

$\det A \neq 0$, use LaSalle's Thm for sufficiency, Chetaev Instability Thm for necessity.

Stability of LQR ~ 2

Now consider, $\dot{x} = Ax + Bu,\ u = Kx \Rightarrow \dot{x} = \left(A + BK\right)x$ where

$$K = -R^{-1}B^T X$$

$$XA + A^T X - XBR^{-1}B^T X = -Q$$

$$\Downarrow \quad +XBK - XB\left(-R^{-1}B^T X\right) + \left(BK\right)^T X - \left(B\left(-R^{-1}B^T X\right)\right)^T X$$

$$X(A + BK) + (A + BK)^T X = -Q - XBR^{-1}B^T X$$

Lyapunov Equation

Generalization of LQR

$$J(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left[x^T(\tau)Qx(\tau)+2x^T(\tau)Su(t)+u^T(\tau)Ru(\tau)\right]d\tau$$

$$R>0,Q-SR^{-1}S\geq 0$$

Notice that (complete the square)

$$x^TQx+2x^TSu+u^TRu=\left(u+R^{-1}Sx\right)^T R\left(u+R^{-1}Sx\right)^T+x^T\left(Q-SR^{-1}S^T\right)x$$

Define a new control

$$\tilde{u}=u+R^{-1}S^Tx,\quad \dot{x}=Ax+Bu\Rightarrow\dot{x}=\left(A-BR^{-1}S^T\right)x+B\tilde{u}$$

$$J(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left[x^T(\tau)\left(Q-SR^{-1}S^T\right)x(\tau)+\tilde{u}^T(\tau)R\tilde{u}(\tau)\right]d\tau$$

$$\Rightarrow\boxed{\begin{array}{c}u=-R^{-1}\left(B^TP+S\right)x\\ P\left(A-BR^{-1}S^T\right)+\left(A-BR^{-1}S^T\right)^T P-PBR^{-1}B^TP+\left(Q-SR^{-1}S^T\right)\end{array}}$$

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RE & Associated Hamiltonian Matrix

$$A^T X + XA - XBR^{-1}B^T X + Q = 0 \Rightarrow H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

- if λ is an eigenvalue of H, so is $-\lambda$
- (A,B) controllable, (A,C) observable where $Q = C^T C$

$\Rightarrow$ there are no eigenvalues on the imaginary axis

$\exists T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ such that $T^{-1}HT = \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix}$

Λ is composed of real Jordan blocks, with $\mathrm{Re}\,\lambda(\Lambda) > 0$

The positive definite solution is $\boxed{X = T_{21}T_{11}^{-1}}$

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Computational Tools

LQR Linear-quadratic regulator design for continuous-time systems.

[K,S,E] = LQR(A,B,Q,R,N) calculates the optimal gain matrix K such that the state-feedback law u = -Kx minimizes the cost function

J = Integral {x'Qx + u'Ru + 2*x'Nu} dt

subject to the state dynamics $\dot{x} = Ax + Bu$.

The matrix N is set to zero when omitted. Also returned are the Riccati equation solution S and the closed-loop eigenvalues E:

$$SA + A'S - (SB+N)R^{-1}(B'S+N') + Q = 0 , \quad E = EIG(A-B*K) .$$

Example: Boeing 747 Longitudinal Dynamics

Consider the longitudinal dynamics of a Boeing 747 at 20,000 ft with a speed of 830 fps

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -0.00643 & 0.0263 & 0 & -32.2 & 0 \\ -0.0941 & -0.624 & 820 & 0 & 0 \\ -0.000222 & -0.00153 & -0.668 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 830 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} 0 \\ -32.7 \\ -2.08 \\ 0 \\ 0 \end{bmatrix} \delta_e, \; y = \theta$$

$$\begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix} \quad \begin{matrix} \text{body x velocity} \\ \text{body z velocity} \\ \text{pitch rate} \\ \text{pitch} \\ \text{altitude} \end{matrix}$$

Boeing 747 Open Loop Modes

```
A=[-0.0064 0.0263 0 -32.2 0;-0.0941 -0.624 761 -196.2 0;
        -.0002 -0.0015 -4.41 -12.48 0;0 0 1 0 0;0 -1 0 830 0];
[V,E]=eigs(A)

V =
   0.0032 + 0.0009i   0.0032 - 0.0009i   0.0252 + 0.0000i   0.0673 + 0.0000i   0.0000 + 0.0000i
   0.9985 + 0.0000i   0.9985 + 0.0000i  -0.4297 + 0.0000i  -0.0104 + 0.0000i   0.0000 + 0.0000i
  -0.0018 + 0.0039i  -0.0018 - 0.0039i  -0.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0011 - 0.0002i   0.0011 + 0.0002i   0.0001 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.0294 + 0.0449i  -0.0294 - 0.0449i  -0.9026 + 0.0000i  -0.9977 + 0.0000i   1.0000 + 0.0000i

E =
  -2.2487 + 2.9839i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -2.2487 - 2.9839i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.5325 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0105 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

Example: Boeing 747 Longitudinal

```
>> A=[-0.0064 0.0263 0 -32.2 0;-0.0941 -0.624 761 -196.2 0;
                    -.0002 -0.0015 -4.41 -12.48 0;0 0 1 0 0;0 -1 0 830 0];
>> b=[0;-32.7;-2.08;0;0];
>> Q = [1,0,0,0,0;0,1,0,0,0;0,0,1,0,0;0,0,0,1,0;0,0,0,0,0.000001];
>> [K,S,E] = lqr(A,b,0.001*Q,1)
K =
    0.0253   -0.0178   -2.3696   -4.1886   -0.0000
E =
  -5.1565 + 5.4556i
  -5.1565 - 5.4556i
  -0.1191 + 0.0932i
  -0.1191 - 0.0932i
  -0.0001 + 0.0000i
```

$-5.1565 \pm j5.4556$	$-2.2487 \pm j2.9839$
$-0.1191 \pm j0.0932$	-0.5325
	-0.0105
-0.0001	0
closed loop	open loop

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Boeing 747 Cont'd

```
>> [V,Eig]=eig(A-b*K)
V =
 0.0019 + 0.0007i    0.0019 - 0.0007i  -0.0371 - 0.0027i  -0.0371 + 0.0027i   0.0010 + 0.0000i
 0.9997 + 0.0000i    0.9997 + 0.0000i   0.0148 - 0.0364i   0.0148 + 0.0364i  -0.0002 + 0.0000i
-0.0052 + 0.0079i   -0.0052 - 0.0079i   0.0000 - 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
 0.0012 - 0.0002i    0.0012 + 0.0002i  -0.0001 + 0.0001i  -0.0001 - 0.0001i  -0.0000 + 0.0000i
-0.0200 + 0.0141i   -0.0200 - 0.0141i   0.9985 + 0.0000i   0.9985 + 0.0000i   1.0000 + 0.0000i
```

Linear Quadratic Gaussian (LQG) Problem

Consider the linear system

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + v$$

where w, v are independent, zero-mean white noise processes

$$E\{w(t)\} = 0,\ E\{v(t)\} = 0$$

having covariances

$$E\{w(t)w^T(\tau)\} = W\delta(t-\tau),\ E\{v(t)v^T(\tau)\} = V\delta(t-\tau),\ E\{w(t)v^T(\tau)\} = 0$$

Find $u(t)$ that minimizes

$$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T\left[x^TQx + u^TRu\right]dt\right\},\ Q = Q^T \geq 0,\ R = R^T > 0$$

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LQG Solution

$$u(t) = K\hat{x}(t),\ K = -R^{-1}B^T P$$

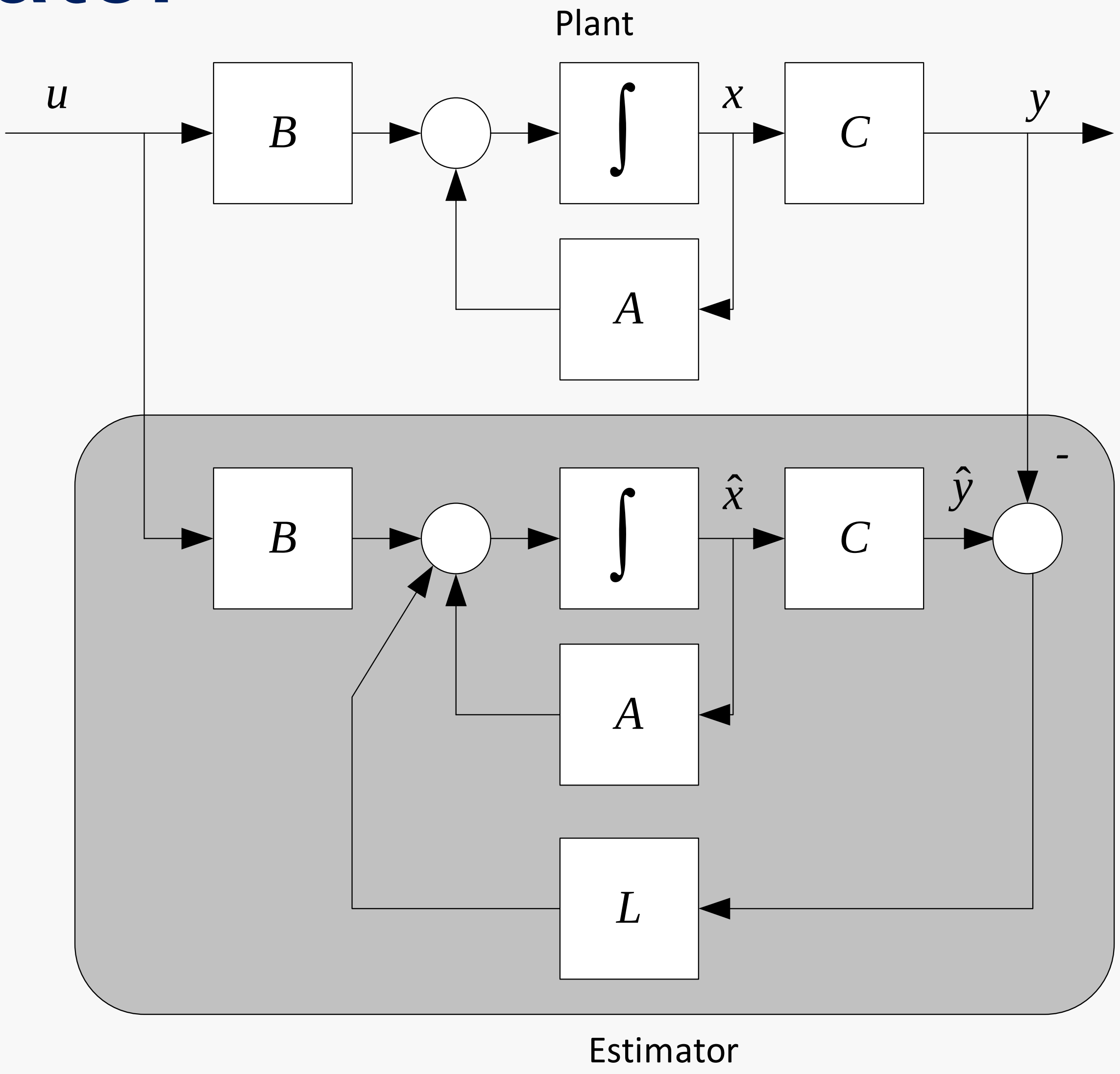
$$PA + A^T P - PBR^{-1}B^T P = -Q,\ P \geq 0$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}),\ L = SC^T V^{-1}$$

$$SA^T + AS - SC^T V^{-1} CS = -W,\ S \geq 0$$

Full-State Estimator

Normally all states are not Available for measurement. Consequently we design a “state estimator” or “observer” which provides estimates based on the measured plant outputs.



Estimator Dynamics

The estimator design approach is to select L by solving the Ricatti Eq

$$SA^T + AS - SC^T V^{-1} CS = -W$$

which compares to

$$PA + A^T P - PBR^{-1} B^T P = -Q,\ P \geq 0$$

with

$$R \Leftrightarrow V,\ Q \Leftrightarrow W,\ B \Leftrightarrow C^T$$